

Lecture 41

1 Scalar and Vector functions

A function f is a *scalar function* if it maps from $\mathbb{R}^n$ to $\mathbb{R}$, i.e., the range of the function is some subset of real number.

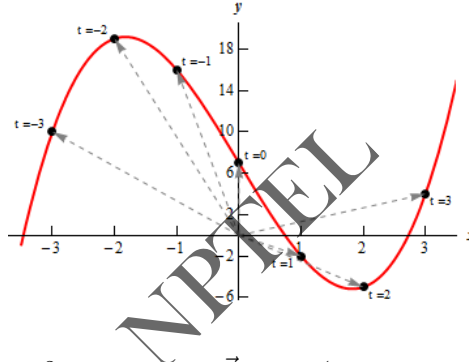
Example 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by (i) $f(x) = x^2$, (ii) $f(x) = e^x$, (iii) $f(x) = \sin x$.

Definition 1. Let $D \subset \mathbb{R}^n$ be a non-empty set. A function $\vec{f} : D \subset \mathbb{R}^n \rightarrow E \subset \mathbb{R}^m$ is called *vector function* if there exist scalar functions $f_1, f_2, \dots, f_m$ defined on D such that

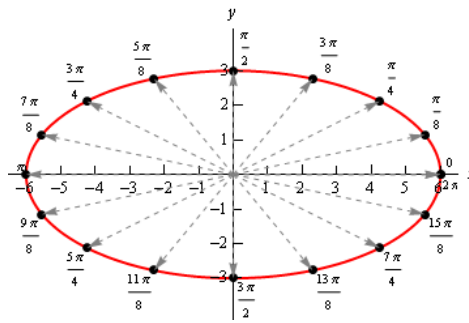
$$\vec{f}(x) = (f_1(x), f_2(x), \dots, f_m(x)).$$

Through out the lecture we consider vector functions which are defined on real line, i.e., $n = 1$.

Example 2. Let $\vec{f} : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $\vec{f}(t) = (t, t^3 - 10t + 7)$. This implies at time t , position of the particle with respect to first coordinate is t and second coordinate is $t^3 - 10t + 7$. The sketch of the vector function is the following.



Example 3. Let $\vec{f} : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $\vec{f}(t) = (6 \cos t, 3 \sin t)$. Value of the $\vec{f}$ at time $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are $(6, 0), (0, 3), (-6, 0), (0, -3)$. The vector function is periodic with the period of 2π . So, in this case the vector function is an ellipse.



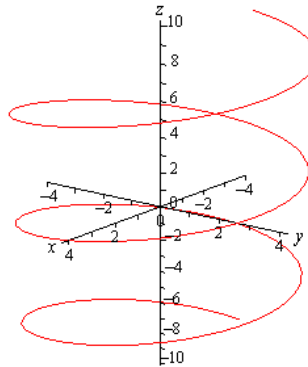
Example 4. The position of a particle on the unit circle at time t can be defined by a vector function $\vec{f}(t) = (\cos t, \sin t)$. This can also be written as

$$\vec{f}(t) = \cos t \hat{i} + \sin t \hat{j}.$$

Similarly, for $m = 3$, the vector function $f : \mathbb{R} \rightarrow \mathbb{R}^3$ is of the form

$$\begin{aligned} \vec{f}(t) &= (4 \cos t, 4 \sin t, t) \\ &= 4 \cos t \hat{i} + 4 \sin t \hat{j} + t \hat{k}. \end{aligned}$$

In this case, along the x and y coordinate function forms a circle. The only change is in the z component and as t increases the z coordinate will increase. So the vector function is forms a spiral.



Scalar field: If to each point $p(x, y, z)$ of a region $\mathcal{R}$ in space there corresponds a unique scalar $f(p)$, then f is called a scalar point function and we say that a scalar field f is defined on $\mathcal{R}$.

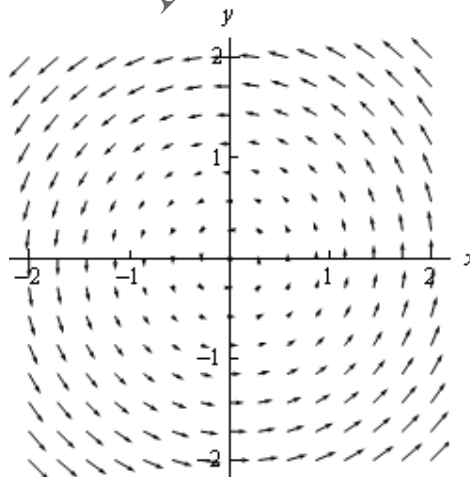
Example: $f(x, y, z) = x^2 + y^2 - 4z^2$ is a scalar field.

Vector field: If to each point $p(x, y, z)$ of a region $\mathcal{S}$ in space there corresponds a unique vector $\vec{f}(p)$, then $\vec{f}$ is called a vector point function and we say that a vector field $\vec{f}$ is defined on $\mathcal{S}$.

Example 5. Let the vector point function $\vec{f}$ on $\mathbb{R}^2$ be defined by

$$\vec{f}(x, y) = -y\hat{i} + x\hat{j}.$$

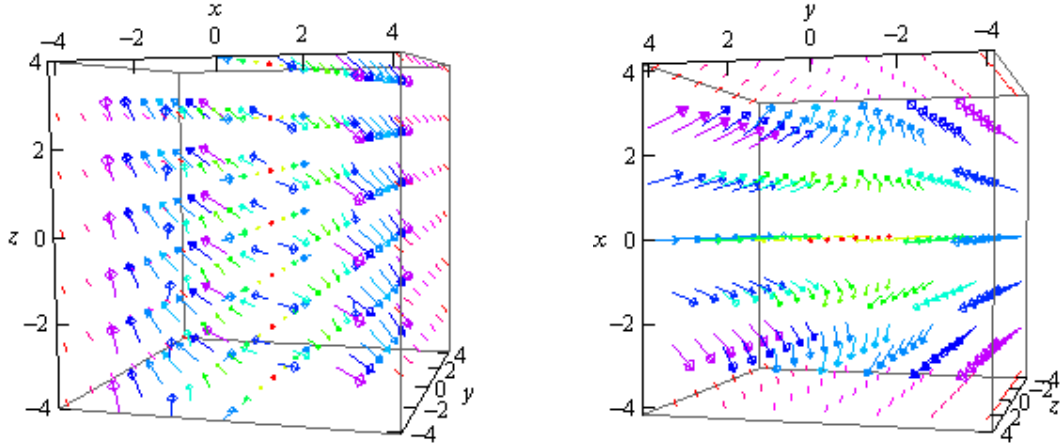
The graph of the vector field is given by the following sketch.



This gives the direction of particle movement at each points in $\mathbb{R}^2$. Similarly, the vector field of the vector function

$$\vec{f}(x, y, z) = 2x\hat{i} - 2y\hat{j} - 2z\hat{k}$$

is given by the following graph.



2 Limit and Continuity

The definitions of limits and continuity for vector functions are simple modifications of the analogous concepts for scalar functions. Using the component functions, we deal with these computations and the idea of continuity in practice.

Definition 2. Let I be an open interval, and let $\vec{f}: I \rightarrow \mathbb{R}^m$ be a vector function defined by $\vec{f}(t) = (f_1(t), f_2(t), \dots, f_m(t))$. The limit of $\vec{f}(t)$ is $\vec{\ell} = (\ell_1, \ell_2, \dots, \ell_m)$ as t approaches to a if for each scalar function f_i , $i = 1, 2, \dots, m$, $\lim_{t \rightarrow a} f_i(t) = \ell_i$. Then we expressed as

$$\lim_{t \rightarrow a} \vec{f}(t) = \left(\lim_{t \rightarrow a} f_1(t), \lim_{t \rightarrow a} f_2(t), \dots, \lim_{t \rightarrow a} f_m(t) \right) = \vec{\ell}.$$

Alternatively, we say $\vec{f}(t)$ approaches to $\vec{\ell}$ as t approaches to a , if for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\vec{f}(t) - \vec{\ell}| < \varepsilon \quad \text{whenever } 0 < |t - a| < \delta.$$

Properties:

1. $\lim_{t \rightarrow a} (\vec{f}(t) \pm \vec{g}(t)) = \lim_{t \rightarrow a} \vec{f}(t) \pm \lim_{t \rightarrow a} \vec{g}(t).$
2. for any $c \in \mathbb{R}$, $\lim_{t \rightarrow a} c \cdot \vec{f}(t) = c \cdot \lim_{t \rightarrow a} \vec{f}(t).$
3. $\lim_{t \rightarrow a} (\vec{f}(t) \cdot \vec{g}(t)) = \lim_{t \rightarrow a} \vec{f}(t) \cdot \lim_{t \rightarrow a} \vec{g}(t).$
4. $\lim_{t \rightarrow a} (\vec{f}(t) \times \vec{g}(t)) = \lim_{t \rightarrow a} \vec{f}(t) \times \lim_{t \rightarrow a} \vec{g}(t).$

Definition 3. Let I be an open interval. A vector function $\vec{f}: I \rightarrow \mathbb{R}^m$ is continuous at a if

- (i) $\vec{f}(a)$ is defined.
- (ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\vec{f}(t) - \vec{f}(a)| < \varepsilon \quad \text{whenever } |t - a| < \delta.$$

Alternatively, $\vec{f}$ is continuous at a , if

$$\lim_{t \rightarrow a} \vec{f}(t) = \vec{f}(a).$$

Example 6. Let $\vec{f}(t) = \left(\frac{\sin t}{t}, t^2 - 3t + 3, \cos t\right)$.

$$\begin{aligned}\lim_{t \rightarrow 0} \vec{f}(t) &= \left(\lim_{t \rightarrow 0} \frac{\sin t}{t}, \lim_{t \rightarrow 0} t^2 - 3t + 3, \lim_{t \rightarrow 0} \cos t\right) \\ &= (1, 3, 1).\end{aligned}$$

While the second and third components of $\vec{f}$ are defined at $t = 0$, the first component $\frac{\sin t}{t}$ is not. Hence, $\vec{f}$ is not defined at $t = 0$. Therefore, $\vec{f}$ is not continuous at $t = 0$. However, limit of $\vec{f}(t)$ exist as t approaches to 0.

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3 Differentiation & Integration

Consider a vector-valued function $\vec{r}$ defined on a open interval I containing t_0 and t_1 , one can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, see Figure 2(a). Dividing the displacement vector by $t_1 - t_0$ gives the average rate of change of $\vec{r}$ on $[t_0, t_1]$, see Figure 2(b).

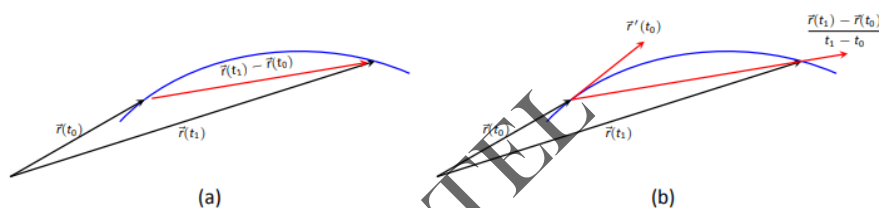


Figure 2: Illustrating displacement, leading to an understanding of the derivative of vector-valued functions.

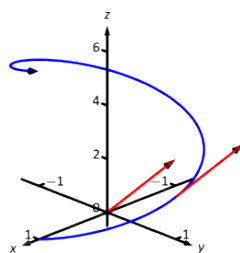
Definition 4. Let $\vec{r}$ be a continuous function on an open interval I containing a . Then the derivative of $\vec{r}$ at $t = a$ is given by

$$\vec{r}'(a) = \left. \frac{d\vec{r}}{dt} \right|_{t=a} = \lim_{h \rightarrow 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}.$$

Let f_1, f_2, f_3 be a differentiable scalar function. If the vector function defined by $\vec{r}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$. Then

$$\vec{r}'(t) = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}.$$

Example 7. Let $\vec{f}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$. Then $\vec{f}'(t) = -\sin t \hat{i} + \cos t \hat{j} + \hat{k}$. At $t = \frac{\pi}{2}$, we have $\vec{f}'(\frac{\pi}{2}) = -\hat{i} + \hat{k}$.



Properties: Let $\vec{f}$ and $\vec{g}$ be differentiable vector functions, let h be a differentiable scalar function, and let c be a real number.

1. $\frac{d}{dt}(\vec{f}(t) \pm \vec{g}(t)) = \vec{f}'(t) \pm \vec{g}'(t).$
2. $\frac{d}{dt}(c\vec{f}(t)) = c\vec{f}'(t).$
3. $\frac{d}{dt}(h(t)\vec{f}(t)) = h'(t)\vec{f}(t) + h(t)\vec{f}'(t).$ **Product rule**
4. $\frac{d}{dt}(\vec{f}(t) \cdot \vec{g}(t)) = \vec{f}'(t) \cdot \vec{g}(t) + \vec{f}(t) \cdot \vec{g}'(t).$ **Product rule**
5. $\frac{d}{dt}(\vec{f}(t) \times \vec{g}(t)) = \vec{f}'(t) \times \vec{g}(t) + \vec{f}(t) \times \vec{g}'(t).$ **Product rule**
6. $\frac{d}{dt}(\vec{f}(h(t))) = \vec{f}'(h(t))h'(t).$ **Chain Rule**

Successive derivative: Let the vector function $\vec{f}$ be n -times differentiable. Then

$$\begin{aligned} \text{1st Order: } & \frac{d\vec{f}}{dt} \\ \text{2nd Order: } & \frac{d^2\vec{f}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{f}}{dt}\right) \\ & \vdots \\ \text{nth Order } & \frac{d^n\vec{f}}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1}\vec{f}}{dt^{n-1}}\right) \end{aligned}$$

Example 8.

1. If a curve is given by $\vec{r}(t) = \vec{a}t + \vec{b}$, then $\frac{d^2\vec{r}}{dt^2} = \vec{0}$.
2. If $\vec{r}(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$. Then

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \hat{i} + (2t+1)\hat{j} + (3t^2+2t+1)\hat{k} \\ \frac{d^2\vec{r}}{dt^2} &= 2\hat{j} + (6t+2)\hat{k} \\ \frac{d^3\vec{r}}{dt^3} &= 6\hat{k} \end{aligned}$$

3. If $\vec{r}(t) = \sin t \hat{i} + \cos t \hat{j} + t \hat{k}$. Then

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \cos t \hat{i} - \sin t \hat{j} + \hat{k} \\ \frac{d^2\vec{r}}{dt^2} &= -\sin t \hat{i} - \cos t \hat{j} \\ \left| \frac{d\vec{r}}{dt} \right| &= \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \\ \left| \frac{d^2\vec{r}}{dt^2} \right| &= \sqrt{\sin^2 t + \cos^2 t} = 1 \end{aligned}$$

4. Let $\vec{r}(t) = \cos nt \hat{i} + \sin nt \hat{j}$. Then $\frac{d\vec{r}}{dt} = -n \sin nt \hat{i} + n \cos nt \hat{j}$.

$$\begin{aligned} \vec{r} \times \frac{d\vec{r}}{dt} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos nt & \sin nt & 0 \\ -n \sin nt & n \cos nt & 0 \end{vmatrix} \\ &= (n \cos^2 nt + n \sin^2 nt) \hat{k} \\ &= n \hat{k}. \end{aligned}$$

Integration: Let $\vec{f}$ and $\vec{F}$ be two vector functions such that $\frac{d\vec{F}}{dt} = \vec{f}$. Then $\vec{F}$ is called the indefinite integration of $\vec{f}$ with respect to t and we write

$$\int \vec{f}(t) dt = \vec{F}(t).$$

Definite integral: $\int_a^b \vec{f}(t) dt = \vec{F}(t)|_{t=a}^{t=b} = \vec{F}(b) - \vec{F}(a).$

Example 9.

1. Let $\vec{f}(t) = e^{2t} \hat{i} + \sin t \hat{j}$. Evaluate $\int_0^1 \vec{f}(t) dt$.

Solution:

$$\begin{aligned} \int_0^1 \vec{f}(t) dt &= \int_0^1 e^{2t} dt \hat{i} + \int_0^1 \sin t dt \hat{j} \\ &= \frac{1}{2} e^{2t} \Big|_0^1 \hat{i} - \cos t \Big|_0^1 \hat{j} \\ &= \frac{1}{2} (e^2 - 1) \hat{i} + (1 - \cos 1) \hat{j} \end{aligned}$$

2. Let $\vec{f}''(t) = 2\hat{i} + \cos t \hat{j} + 12t \hat{k}$. Find $\vec{f}$ where $\vec{f}(0) = -7\hat{i} - \hat{j} + 2\hat{k}$, and $\vec{f}'(0) = 5\hat{i} + 3\hat{j}$.

Solution:

$$\begin{aligned} \vec{f}'(t) &= \int \vec{f}''(t) dt \\ &= 2t \hat{i} + \sin t \hat{j} + 6t^2 \hat{k} + \vec{a}. \\ \vec{f}'(0) &= \vec{a}. \end{aligned}$$

Therefore, $\vec{f}'(t) = (2t + 5) \hat{i} + (\sin t + 3) \hat{j} + 6t^2 \hat{k}$.

$$\begin{aligned} \vec{f}(t) &= \int \vec{f}'(t) dt \\ &= (t^2 + 5t) \hat{i} + (-\cos t + 3t) \hat{j} + 2t^3 \hat{k} + \vec{b}. \\ \vec{f}(0) &= -\hat{j} + \vec{b} \\ \vec{b} &= -7\hat{i} + 2\hat{k}. \end{aligned}$$

Therefore, $\vec{f}(t) = (t^2 + 5t - 7) \hat{i} + (-\cos t + 3t) \hat{j} + (2t^3 + 2) \hat{k}$.

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4 Partial Derivative

Suppose $\vec{r}$ is a vector function depending on more than one scalar variable. Let $\vec{r} = \vec{f}(x, y, z)$, i.e., $\vec{r}$ is a function of variable x , y and z . Thus partial derivative of $\vec{r}$ with respect to x is defined by

$$\frac{\partial \vec{r}}{\partial x} = \lim_{h \rightarrow 0} \frac{\vec{r}(x+h, y, z) - \vec{r}(x, y, z)}{h}.$$

Similarly, we can define $\frac{\partial \vec{r}}{\partial y}, \frac{\partial \vec{r}}{\partial z}, \frac{\partial^2 \vec{r}}{\partial x \partial y}, \dots$

Vector Differential Operator: The vector differential operator $\vec{\nabla}$ (nabla) and it is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}.$$

Gradient of a scalar function: Let f be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f is denoted by

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}.$$

Example 10. Let the scalar function $f(x, y, z) = x^3 y z^2$. Then the gradient of f is

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= 3x^2 y z^2 \hat{i} + x^3 z^2 \hat{j} + 2x^3 y z \hat{k} \\ \vec{\nabla} f(1, 1, 2) &= 12\hat{i} + 4\hat{j} + 4\hat{k}.\end{aligned}$$

Properties: Let f and g be a multi-variable scalar function.

1. $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$
2. $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$

Definition 5. Let $f(x, y, z)$ be a scalar field over a region $\mathcal{R}$. Then the points satisfying an equation of the type

$$f(x, y, z) = c$$

constitute a family of surface in 3-dimensional space is called *level surface*.

Lemma 4.1. Let f be a scalar function. Then $\vec{\nabla} f$ is a vector normal to the surface $f(x, y, z) = c$, where c is a constant.

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Directional derivative: Let $f(x, y, z)$ be a scalar function defined on a region $\mathcal{R}$. Let P be any arbitrary point in $\mathcal{R}$ and suppose Q is a point in the region $\mathcal{R}$ which is a neighboring point of P in the direction of a given unit vector $\hat{a}$. Then the limit $\lim_{Q \rightarrow P} \frac{f(Q) - f(P)}{PQ}$, if it exists, is called directional derivative of f at the point P in the direction of $\hat{a}$.

Remark 4.2. Let $P = (x, y, z)$ and $Q = (x + \delta x, y + \delta y, z + \delta z)$ be two points in the region $\mathcal{R}$. Let $\hat{a}$ be the unit vector and $\delta f = f(Q) - f(P)$. Then $\frac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{a}$. Now, the directional derivative of f at P in the direction of $\hat{a}$ is

$$\lim_{Q \rightarrow P} \frac{f(Q) - f(P)}{PQ} = \lim_{\delta s \rightarrow 0} \frac{\delta f}{\delta s} = \frac{df}{ds}.$$

Theorem 4.3. The directional derivative of a scalar function f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{a}$ is given by

$$\frac{df}{ds} = \vec{\nabla} f \cdot \hat{a}.$$

Example 11. Let $f(x, y, z) = x^3 y z + 4x z^2$ be a scalar function. The directional derivative of f in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$ at a point $(1, -2, -1)$ is given by

$$\left. \frac{df}{ds} \right|_{(1, -2, -1)} = \vec{\nabla} f(1, -2, -1) \cdot \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}.$$

$\vec{\nabla} f(x, y, z) = (3x^2 y z + 4z^2) \hat{i} + x^3 z \hat{j} + (x^3 y + 8xz) \hat{k}$. Therefore, $\vec{\nabla} f(1, -2, -1) = 10\hat{i} - \hat{j} - 10\hat{k}$, and

$$\frac{df}{ds} = \frac{20 + 1 + 20}{3} = \frac{41}{3}.$$

Let $f(x, y, z) = c$ be the equation of the level surface. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a position vector of any point P on this surface. Then $\vec{\nabla}f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ is a vector along the normal to the surface at the point P , i.e., $\vec{\nabla}f$ is perpendicular to the tangent plane at the point P .

Let $Q = (X, Y, Z)$ then $\vec{PQ} = \vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies on the tangent plane at P to the surface. But since, $\vec{\nabla}f$ is perpendicular to the tangent plane, i.e., $\vec{\nabla}f \perp \vec{PQ}$ implies

$$\begin{aligned}\vec{\nabla}f \cdot \vec{PQ} &= 0 \\ (X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} + (Z - z)\frac{\partial f}{\partial z} &= 0.\end{aligned}$$

This is the required equation of a tangent plane at P .

Example 12. Let $x^2y + 2xz = 4$ be the level surface. Unit normal to the level surface at $(2, -2, 3)$ is $\vec{\nabla}f(2, -2, 3)$.

$$\begin{aligned}\vec{\nabla}f(x, y, z) &= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k} \\ \vec{\nabla}f(2, -2, 3) &= -2\hat{i} + 4\hat{j} + 4\hat{k}\end{aligned}$$

Therefore unit normal to the given level surface at $(2, -2, 3)$ is $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$.

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Example 13. In what direction from the point $(1, 1, -1)$ is the directional derivative of $f(x, y, z) = x^2 - 2y^2 + 4z^2$ a maximum? Also find the value of maximum directional derivative.

Solution: We are given that $f(x, y, z) = x^2 - 2y^2 + 4z^2$. $\vec{\nabla}f(x, y, z) = 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$.

The directional derivative of f is maximum in the direction of $\vec{\nabla}f$. Since the point P is given by $(1, 1, -1)$, therefore $\vec{\nabla}f(1, 1, -1) = 2\hat{i} - 4\hat{j} - 8\hat{k} = \vec{a}$.

The maximum directional derivative is given by

$$\frac{df}{ds} = \vec{\nabla}f(1, 1, -1) \cdot \hat{a} = \frac{|\vec{\nabla}f|^2}{|\vec{\nabla}f|} = |\vec{\nabla}f| = 2\sqrt{21}.$$

Example 14. For the function $f(x, y) = \frac{y}{x^2 + y^2}$. Find the value of the directional derivative making an angle 30° with the positive x -axis at the point $(0, 1)$.

Solution: The directional derivative is given by

$$\begin{aligned}\vec{\nabla}(x, y) &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \\ &= -\frac{2xy}{(x^2 + y^2)^2}\hat{i} + \frac{x^2 - y^2}{x^2 + y^2}\hat{j} \\ \vec{\nabla}(0, 1) &= -\hat{j}.\end{aligned}$$

The unit vector which makes an angle 30° with the positive x -axis is given by $\cos 30^\circ\hat{i} + \sin 30^\circ\hat{j}$. Hence the corresponding directional derivative is

$$\vec{\nabla}f(0, 1) \cdot (\cos 30^\circ\hat{i} + \sin 30^\circ\hat{j}) = -\sin 30^\circ = -\frac{1}{2}.$$

Example 15. What is the greatest rate of increase of $u = xyz^2$ at $(1, 0, 3)$?

Solution: $\vec{\nabla}u(x, y, z) = yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$. Hence $\vec{\nabla}u(1, 0, 3) = 9\hat{j}$.

The greatest rate of increase of u at $(1, 0, 3)$ = the maximum value of $\left.\frac{df}{ds}\right|_{(1,0,3)} = |\vec{\nabla}u(1, 0, 3)| = 9$.

Example 16. Find the equation of the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.

Solution: The given surface $f(x, y, z) = 2xz^2 - 3xy - 4x - 7 = 0$. Then

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= (2z^2 - 3y - 4)\hat{i} - 3x\hat{j} + 4xz\hat{k} \\ \vec{\nabla} f(1, -1, 2) &= 7\hat{i} - 3\hat{j} + 8\hat{k}\end{aligned}$$

Here $7\hat{i} - 3\hat{j} + 8\hat{k}$ is the vector along the normal to the surface at $(1, -1, 2)$.

If $R = (X, Y, Z)$ is the position vector of any point in tangent plane at $(1, -1, 2)$. Then the vector $\vec{R} - (\hat{i} - \hat{j} + 2\hat{k})$ is perpendicular to the vector $\vec{\nabla} f(1, -1, 2)$. Therefore, the required equation of tangent is

$$\begin{aligned}((X - 1)\hat{i} + (Y + 1)\hat{j} + (Z - 2)\hat{k}) \cdot \vec{\nabla} f(1, -1, 2) &= 0 \\ 7(X - 1) - 3(Y + 1) + 8(Z - 2) &= 0 \\ 7X - 3Y + 8Z &= 20.\end{aligned}$$

The equation of the normal to the surface at the point $(1, -1, 2)$ is

$$\begin{aligned}\frac{X - 1}{\frac{\partial f}{\partial x}} &= \frac{Y + 1}{\frac{\partial f}{\partial y}} = \frac{Z - 2}{\frac{\partial f}{\partial z}} \\ \frac{X - 1}{7} &= \frac{Y + 1}{-3} = \frac{Z - 2}{8}.\end{aligned}$$

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